# A MEREOLOGICAL LOGIC OF PRIME NUMBERS: RECURSIVE FOUNDATIONS AND PHILOSOPHICAL IMPLICATIONS

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Abstract. This paper develops a formal mereological framework for generating prime numbers through recursive construction from part–whole relationships. We define a sequence $P = \{P_n : n \in \mathbb{N}\}$ with $P_1 = 1$ that generates all natural numbers through finite products, where primality emerges from structural irreducibility within the recursive framework. While the modern convention excludes 1 to preserve unique factorization, our recursive approach incorporates it as a foundational element without loss of equivalence for $n \geq 2$. The framework yields $\{1, 2, 3, 5, 7, 11, \ldots\}$ while aligning with classical number theory beyond the first element. This generative account distinguishes between backward–looking divisibility tests and forward–looking recursive construction, offering a complementary perspective on prime foundations. The approach extends systematically to the real numbers, demonstrating that alternative recursive axiomatizations can preserve mathematical rigor while revealing structural insights into mathematical foundations.

## 1. Introduction

The definition and classification of prime numbers have evolved over mathematical history, reflecting both computational needs and theoretical developments. While Euclid's *Elements* [1]established fundamental properties of primes, the question of whether 1 should be considered prime remained debated into the early 20th century. The modern convention excludes 1 to preserve unique factorization, but this exclusion may be interpreted as a convention rather than a structural necessity.

The modern mathematical convention, which excludes 1 from the primes, emerged primarily to preserve the elegant statement of the Fundamental Theorem of Arithmetic and to avoid systematic exceptions in various number-theoretic results. This conventional choice has

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proven mathematically fruitful, enabling clean formulations of important theorems and supporting the development of abstract algebra and number theory.

This paper explores an alternative approach with recursive construction that naturally incorporates 1 as a foundational element. Rather than challenging established mathematical practice, we investigate how different foundational choices can preserve mathematical content while revealing alternative structural relationships. Our recursive framework demonstrates complete equivalence with classical prime theory for all $n \geq 2$ while providing additional insights into the generative structure underlying arithmetic.

We develop a formal mereological approach that reconceptualizes prime generation through recursive construction based on part-whole relationships. drawing on philosophical mereology. This perspective, drawing on philosophical work and inspiration by [2], [3], and [4], while remaining faithful to mathematical practice. Rather than challenging established mathematical conventions, we aim to provide deeper philosophical understanding of why these conventions arise and what alternative conceptualizations might offer to both mathematics and philosophy. The framework naturally incorporates recent developments in mathematical structuralism, [6] and [7] by treating mathematical objects as positions within recursively generated structures rather than entities defined solely through external relations.

Our approach centers on a recursive definition where $\mathcal{P} = \{P_n : n \in \mathbb{N}\}$ generates all natural numbers through finite products, with $P_1 = 1$ serving as the necessary foundation for complete arithmetic construction. We demonstrate that this recursive approach maintains full compatibility with traditional number theory while providing additional structural insights about mathematical foundations. Where divisor-count definitions are backward-looking (testing membership), the recursive definition is forward-generative (constructing the sequence). It is through this generative approach that 1's structural necessity becomes apparent.

This framework contributes to ongoing discussions in philosophy of mathematics and foundational studies by demonstrating how alternative recursive constructions can preserve mathematical utility while revealing different structural perspectives. The recursive framework shows complete equivalence with classical prime theory for all $n > 1$ while illuminating the structural role of foundational elements in systematic mathematical construction.

## 2. Mereology and Prime Numbers

### 2.1. Formal Mereological Framework for Prime Generation.

We introduce a new definition based on mereological principles—the philosophical study of part-whole relationships [2, 3, 4].

**Definition 2.1** (Recursive Prime Generation)**.** Let $\mathcal{P} = \{P_n : n \in \mathbb{N}\}$ be defined recursively as follows:

(1) Base case: $P_1 = 1$.
(2) Recursive step: For $n \geq 1$,

$$P_{n+1} = \min \{k \in \mathbb{N} : k > P_n \wedge k \notin \mathcal{C}_n\}$$

where $\mathcal{C}_n = \{a \cdot b : a, b \in \text{span}(\{P_i\}_{i=1}^n), a > 1, b > 1\}$ and $\text{span}(\{P_i\}_{i=1}^n)$ denotes the set of all finite products of elements from $\{P_1, P_2, \ldots, P_n\}$.

#### 2.1.1. *Philosophical Foundations of the Mereological Approach.*

Choice of mereological logic as our foundational framework reflects deeper commitments about the nature of mathematical objects and their relationships. Traditional approaches to prime theory begin with divisibility relations—asking whether one number divides another—which presupposes the independent existence of both numbers as completed entities. This approach treats mathematical objects as self-subsistent items that subsequently enter into external relations.

Mereological logic inverts this priority by treating mathematical objects as fundamentally constituted through part-whole relationships [2, 3]. In this framework, a number's mathematical identity emerges from its position within a compositional hierarchy rather than from intrinsic properties it possesses independently of that hierarchy. The recursive construction $\mathcal{P} = \{P_n : n \in \mathbb{N}\}$ embodies this principle: each element $P_n$ exists not as a discovered entity but as a structural position necessitated by the system's compositional requirements.

This philosophical reorientation has concrete mathematical consequences. Where divisibility-based definitions ask "What properties must an object have to count as prime?", the mereological approach asks "What structural role must the next element play to extend the system's generative capacity?" The former treats primality as a property that numbers either possess or lack; the latter reveals primality as a structural function within recursive construction processes.

The foundational status of $P_1 = 1$ exemplifies this distinction. Traditional approaches exclude 1 from primality to preserve unique factorization, treating this exclusion as a necessary exception to general principles. The mereological framework reveals instead that 1 occupies a unique generative role—not as an exception to primality but as the structural foundation that makes primality possible. This shifts the philosophical question from "Why should 1 be excluded?" to "What foundational role must 1 play in any complete constructive system?"

This mereological perspective treats numbers as structured entities with part-whole relationships rather than merely quantities or abstract objects. In traditional mereology, an atomic part is one that cannot be further divided within the system. Similarly, prime numbers in our framework are atomic in that they cannot be constructed from previously identified primes [2].

**Theorem 2.2** (Completeness of Natural Number Generation)**.** *Every natural number $n \in \mathbb{N}$ can be expressed as a finite product of elements from $\mathcal{P}$.*

*Proof.* We prove by strong induction on $n$.

*Base case:* $n = 1 = P_1$ is trivially a product of elements from $\mathcal{P}$.

*Inductive step:* Assume the statement holds for all $k < n$. If $n \in \mathcal{P}$, then $n$ is already a product (of itself). If $n \notin \mathcal{P}$, then by construction of the sequence $\mathcal{P}$, there exist $a, b \in \text{span}(\{P_i : P_i < n\})$ with $a, b > 1$ such that $n = a \cdot b$. By the inductive hypothesis, both $a$ and $b$ can be expressed as products of elements from $\mathcal{P}$, hence so can $n$. □

**Theorem 2.3** (Necessity of $P_1 = 1$)**.** *For the recursive construction to generate all natural numbers, we must have $P_1 = 1$.*

*Proof.* Suppose $P_1 = k > 1$. Then the smallest product of two elements from $\mathcal{P}$ would be $k^2$, which is greater than all natural numbers in $\{1, 2, \ldots, k^2 - 1\}$. By the recursive construction, these numbers cannot be expressed as non-trivial products, so they would all need to be in $\mathcal{P}$. But this contradicts the minimality condition in the recursive step, as there would be multiple candidates for $P_2$. Therefore, $P_1 = 1$ is necessary for the construction to cover all natural numbers. □

**Theorem 2.4** (Equivalence with Traditional Primes)**.** *For $n \geq 2$, $P_n$ is prime in the traditional sense if and only if $P_n \in \mathcal{P}$ with $P_n > 1$.*

*Proof.* Let $p = P_n$ with $n \geq 2$, so $p > 1$.

($\Rightarrow$) Suppose $p$ is traditionally prime. If $p \notin \mathcal{P}$, then $p \in \mathcal{C}_k$ for some $k$, meaning $p = a \cdot b$ where $a, b \in \text{span}(\{P_i\}_{i=1}^k)$ and $a, b > 1$. This implies $p$ has non-trivial divisors, contradicting traditional primality.

($\Leftarrow$) Suppose $p \in \mathcal{P}$ with $p > 1$. By construction, $p \notin \mathcal{C}_{n-1}$, so $p$ cannot be expressed as a product $a \cdot b$ with $a, b > 1$ from prior elements. This means $p$ has no non-trivial divisors other than 1 and itself, hence $p$ is traditionally prime. □

**Lemma 2.5** (Equivalence to Standard Prime Sequence). *Let* $\mathcal{P} = \{P_n\}_{n\in\mathbb{N}}$ *be defined recursively as in Definition 1, with* $P_1 = 1$*, and let* $\mathbb{P}' = \{p_n\}_{n\geq 1}$ *denote the standard increasing sequence of prime numbers starting with* $p_1 = 2$*. Then:*

$$\{P_n\}_{n\geq 2} = \{p_n\}_{n\geq 1}$$

*That is, the recursive construction generates the standard prime sequence beginning at 2, with* $P_1 = 1$ *functioning only as the multiplicative identity and never participating in* $\mathcal{C}_n$*.*

The foundational position in the generative structure of natural numbers was originally formalized by Dedekind [15] and Peano [8] through recursive successor axioms. This position was later elaborated conceptually in structural terms by Penrose [9], and it grounds our mereological formulation.

*Proof.* We proceed by induction [5, 8]:

*Base Case:* $P_2$

From Definition 1, we have:

(1) $P_1 = 1$.
(2) $\mathcal{C}_1 = \{a \cdot b \mid a, b \in \mathcal{M}_1,\ a > 1,\ b > 1\}$.

**Theorem 2.6** (Well-definedness). *The sequence* $\{P_n\}$ *is infinite and uniquely defined.*

*Proof.* We proceed by strong induction on $n$.

*Base case:* $P_1 = 1$ is well-defined by definition.

*Inductive step:* Assume $P_1, P_2, \ldots, P_n$ are well-defined. The set $\mathcal{C}_n$ is finite since it consists of products of finitely many finite sets. The set $\{k \in \mathbb{N} : k > P_n \wedge k \notin \mathcal{C}_n\}$ is non-empty because, by Euclid's argument, the number $1 + \prod_{i=2}^{n} P_i$ is greater than $P_n$ and cannot be expressed as a product of elements from $\{P_2, \ldots, P_n\}$ with factors greater than 1. Therefore, $P_{n+1}$ exists and is unique by the well-ordering principle. ☐

But:

(1) $\mathcal{M}_1 = \{1^k = 1 \mid k \geq 1\} = \{1\}$.
(2) Since all elements $a, b > 1$ are excluded, $\mathcal{C}_1 = \emptyset$.

Thus:

$$P_2 = \min\{x > 1 \mid x \notin \emptyset\} = 2 = p_1$$

*Inductive Step:* Assume $\{P_2, \ldots, P_n\} = \{p_1, \ldots, p_{n-1}\}$. Then:

(1) $\mathcal{M}_n$ is the set of all finite products of $\{1, p_1, \ldots, p_{n-1}\}$.
(2) $\mathcal{C}_n$ is the set of all composite numbers formed from those primes (excluding 1 as a factor).

Since each $P_k$ is chosen as the minimal integer not expressible as a product of smaller $P_i$'s, and since all composites are contained in $\mathcal{C}_n$, the next $P_{n+1}$ is the next integer not divisible by any smaller $P_i$, i.e., the next prime.

Thus: $P_{n+1} = p_n$

By the principle of mathematical induction, the sequence $\{P_n\}_{n\geq 2}$ coincides with the standard prime sequence [9], $\{p_n\}_{n\geq 1}$. □

**Corollary 2.7** (Unique Mereological Decomposition)**.** *Then every natural number $m \geq 2$ admits a unique multiset decomposition as a finite product [10].:*

$$m = \prod_{j=1}^{k} P_{i_j}$$

*where each $P_{i_j} \in \mathcal{P} \setminus \{1\}$, and the multiset $\{P_{i_1}, \ldots, P_{i_k}\}$ is uniquely determined up to permutation.*

**Interpretation (Mereological Perspective).** This corollary recasts the classical Fundamental Theorem of Arithmetic in terms of parts and construction:

(1) A natural number $m$ is composed from irreducible parts (the prime numbers $P_n$, with $P_1 = 1$ excluded from participation in composite generation).
(2) Each number greater than 1 is built from a unique collection of irreducible constituents, viewed here not as a tuple of exponents (as in classical prime factorization), but as a multiset of parts from the recursive generator.

**Proof Sketch.** From Lemma 1, the sequence $\{P_n\}_{n\geq 2}$ coincides with the standard primes $\{p_n\}$, which are known to generate all integers $\geq 2$ via unique prime factorizations.

Since:

(1) $\mathcal{M}_n$ (the multiplicative closure of $\{P_k\}_{k\leq n}$) is exactly the set of positive integers generated by products of these primes.
(2) No element of $\mathcal{P}\setminus\{1\}$ is itself a product of smaller such elements.

Then: Every integer $m \geq 1$ can be uniquely written as a product of these irreducibles, which is a mereological decomposition into irreducible parts, ordered as a multiset. □

2.2. **Relationship to Classical Prime Theory.** [Structural Relationship to Classical Theory] The recursive construction $\mathcal{P}$ satisfies the following relationships with classical prime theory:

(1) **Sequence Correspondence:** $\{P_n : n \geq 2\} = \{p \in \mathbb{N} : p \text{ is classically prime}\}$

(2) **Factorization Preservation:** For any $n > 1$, the unique prime factorization of $n$ in classical theory corresponds exactly to its representation as a product of elements from $\{P_k : k \geq 2\}$
(3) **Foundational Extension:** The recursive framework extends classical theory by providing the generative foundation $P_1 = 1$ while preserving all classical results

*Proof.* Statement (1) follows directly from Theorem 5. Statement (2) follows from the construction ensuring that products involving only elements from $\{P_k : k \geq 2\}$ correspond to classical factorizations. Statement (3) is established by Theorems 3 and 4 showing necessity and completeness of the foundational extension. □

**Insight:** This framework reveals that classical prime theory captures the irreducible elements for $n > 1$ while the recursive construction exposes the underlying generative structure that classical theory presupposes but does not formalize.

2.3. **Recursive Structure and Asymptotic Properties.** The recursive framework generates prime structure dynamically through well-defined construction rules. This contrasts with classical approaches defining primality through divisibility testing, instead revealing primes as emergent consequences of structural irreducibility [11].

*Example* 2.8 (Construction Sequence). The recursive generation proceeds as follows:

(1) $P_1 = 1$ (base case).
(2) $P_2 = \min\{k > 1 : k \notin \{1 \cdot 1\}\} = 2$ (since products with factors $> 1$ from $\{1\}$ yield an empty set).
(3) $P_3 = \min\{k > 2 : k \notin \{1 \cdot 1, 2 \cdot 2\}\} = 3$.
(4) $P_4 = \min\{k > 3 : k \notin \{1 \cdot 1, 2 \cdot 2, 2 \cdot 3, 3 \cdot 2\}\} = 5$ (since $4 = 2 \cdot 2$).
(5) $P_5 = \min\{k > 5 : k \notin \mathcal{C}_4\} = 7$ (since $6 = 2 \cdot 3$).

yielding the sequence $\mathcal{P} = \{1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \ldots\}$.

**Theorem 2.9** (Asymptotic Density)**.** *Let* $\pi_{\mathcal{P}}(x) = |\{P_n \in \mathcal{P} : P_n \leq x, n \geq 2\}|$ *denote the counting function for elements of* $\mathcal{P}$ *greater than 1. Then*

$$\pi_{\mathcal{P}}(x) \sim \frac{x}{\ln x} \text{ as } x \to \infty$$

*Proof.* By Theorem 5, $\{P_n : n \geq 2\} = \{p : p \text{ classically prime}\}$. Therefore, $\pi_{\mathcal{P}}(x) = \pi(x)$ where $\pi(x)$ is the classical prime counting function. The asymptotic formula follows from the Prime Number Theorem [11]. □

**Theorem 2.10** (Infinitude of $\mathcal{P}$)**.** *The sequence* $\mathcal{P}$ *is infinite.*

*Proof.* Suppose $\mathcal{P}$ is finite with $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$. Consider the number $Q = 1 + \prod_{i=2}^{k} P_i$. Note that $Q > P_k$ and $Q \equiv 1 \pmod{P_i}$ for all $i \geq 2$. Therefore, $Q$ cannot be expressed as a product of elements from $\{P_2, \ldots, P_k\}$ with factors greater than 1, implying $Q \notin \mathcal{C}_k$. By the recursive construction, there exists $P_{k+1} \leq Q$, contradicting the assumption that $\mathcal{P}$ has only $k$ elements. □

2.4. **The Generative Nature of Recursive Construction.** The recursive framework developed in Definitions 1–2 embodies a fundamentally different approach to mathematical construction than traditional recursive procedures. While conventional recursion operates through backward reduction—defining ramified cases in terms of simpler instances of the same structure—our mereological construction employs what we term *generative recursion*, where mathematical structure emerges through forward-directed structural necessity.

2.4.1. *Forward vs. Backward Recursion.* Traditional recursive definitions follow the pattern $f(n) = g(f(n-1))$, where intricate cases are computed by reduction to previously established instances. This approach, rooted in Dedekind's foundational work on the nature of number and recursive definition [12], treats the base case as a computational terminus rather than a generative origin. In contrast, our construction operates through forward structural extension:

(1) **Traditional Recursion:** Given $f(1)$, compute $f(2), f(3), \ldots$ by applying reduction rules.
(2) **Generative Recursion:** From $P_1 = 1$, each $P_{n+1}$ emerges as the next structurally necessary extension.

The crucial difference lies in temporal and logical direction. Traditional recursion asks "What simpler case leads to this result?" Our framework asks "What must come next to maintain structural integrity?"

Consider how $P_1 = 1$ functions in our system. Rather than serving as a mere base case for computational termination, 1 acts as the generative seed that initiates the entire structural unfolding. $P_2 = 2$ emerges not because it reduces to $P_1$, but because 2 represents the first number that cannot be constructed from $\{1\}$ under our compositional constraints. This forward momentum continues: each subsequent prime emerges as a structural innovation required by the system's internal logic.

2.4.2. *Structural Necessity vs. Computational Procedure.* The heart of generative recursion lies in *structural necessity*—the logical requirement that certain mathematical objects must emerge to preserve the system's constructive capacity. This differs fundamentally from computational procedures that calculate predetermined sequences.

In traditional approaches, primality is determined through divisibility testing: "Is $p$ divisible by any number less than itself?" This reactive procedure identifies primes within a pre-existing number system. Our framework instead asks: "What number must emerge next to extend the system's generative capacity?"

Each prime in our construction represents a structural innovation—a number that adds new compositional possibilities that cannot be achieved through existing elements. $P_3 = 3$ emerges not because it passes a divisibility test, but because no combination of $\{1, 2\}$ can generate 3. The system requires 3 as a new generative element to continue its structural development.

This necessity is logical rather than computational. Traditional recursion computes what already exists in principle; generative recursion discovers what must exist for logical consistency. The difference parallels the distinction between calculating solutions to known equations versus deriving the equations themselves from fundamental principles.

2.4.3. *Mathematical Innovation Through Recursive Generation.* Generative recursion influences how mathematical structure can be self-organizing through internal logical constraints. Rather than imposing external definitions (such as "primes are numbers with exactly two divisors"), the system generates its own structural categories through recursive self-extension.

This process exhibits three key characteristics:

(1) **Self-Determination:** Each step in the construction is determined by the system's internal state rather than external criteria. $P_{n+1}$ emerges from the structural requirements imposed by $\{P_1, \ldots, P_n\}$, not from independent testing procedures.
(2) **Irreversible Innovation:** Once a prime emerges, it permanently expands the system's generative capacity. Unlike computational recursion, which can retrace its steps, generative recursion creates irreversible structural novelty.
(3) **Logical Inevitability:** The sequence that emerges exhibits logical necessity—given the starting conditions and construction rules, the specific sequence $\{1, 2, 3, 5, 7, 11, \ldots\}$ represents the unique path of structural development.

This perspective illuminates why our construction naturally incorporates 1 without special exclusion. In backward-looking recursion, 1 appears problematic because it disrupts computational patterns. In forward-looking generation, 1 is logically necessary as the unique starting point that can initiate structural development without presupposing prior elements.

The philosophical implications extend beyond number theory. Generative recursion suggests a general principle for understanding how mathematical structures emerge from simple foundations through logical self-organization rather than external imposition. This framework may illuminate similar patterns in other domains where hierarchical structure emerges through recursive construction processes—from formal logical systems to computational complexity theory to the organization of deterministic physical systems.

The recognition that mathematical objects can emerge through generative rather than reductive recursion opens new perspectives on mathematical foundations, suggesting that structural necessity may be more fundamental than definitional stipulation in determining the logical architecture of mathematical systems.

## 3. Mathematical Necessity of the Foundational Element

### 3.1. Structural Characterization of the Foundational Element.

The recursive construction reveals that $P_1 = 1$ possesses unique structural properties analogous to ground states in physical systems. While traditional number theory excludes 1 from primality through definitional convention, our framework demonstrates that 1 occupies a mathematically necessary foundational position in the generative structure of natural numbers [8].

Ontologically, this interpretation aligns with Leśniewski's part–whole logic [3], where prime 1 functions as the atomic part that grounds arithmetic construction. [2] analysis of ontological atomicity further supports this reading: although he does not address number theory directly, his treatment of indivisible parts as structurally generative justifies our interpretation of 1 as the irreducible root of the number system. In this light, 1 is not merely the multiplicative identity but the ontological ground state of the number-theoretic hierarchy—a necessary structural origin from which all prime composition unfolds.

**Theorem 3.1** (Foundational Necessity). *In the recursive construction $\mathcal{P}$, the element $P_1 = 1$ is uniquely characterized as the minimal element with respect to the generative ordering, and no alternative choice for $P_1$ yields a complete generation of $\mathbb{N}$.*

*Proof.* We establish necessity through contradiction and uniqueness through minimality.

*Necessity:* Suppose $P_1 = k$ for some $k \neq 1$.

(1) If $k = 0$, then products involving $P_1$ yield 0, failing to generate positive integers.
(2) If $k > 1$, then the minimal non-trivial product is $k^2 > k$, leaving the interval $(1, k^2)$ ungeneratable by products, violating completeness (Theorem 3).
(3) If $k < 0$, products may yield negative results, failing to generate $\mathbb{N}$.

*Uniqueness:* The choice $P_1 = 1$ satisfies the multiplicative identity property, ensuring that $1 \cdot n = n$ for all $n \in \mathbb{N}$, making it the unique minimal element that preserves all natural numbers under multiplication. □

### 3.2. **Structural Properties and Mathematical Implications.**

**Corollary 3.2** (Structural Distinction). *The element $P_1 = 1$ is distinguished from all other elements in $\mathcal{P}$ by the following properties:*

(1) ***Multiplicative neutrality:*** $1 \cdot P_n = P_n$ *for all* $n \in \mathbb{N}$
(2) ***Generative foundation:*** *All elements $P_n$ with $n > 1$ require $P_1$ for complete natural number generation*
(3) ***Irreducible atomicity:*** *$P_1$ cannot be expressed as a non-trivial product within the framework*

*Proof.* Property (1) follows from the definition of multiplication. Property (2) follows from Theorem 4. Property (3) follows from the recursive construction where $\mathcal{C}_0 = \emptyset$, so $P_1$ has no predecessors for decomposition. □

### 3.3. **Factorization Correspondence.** [Factorization Correspondence]

For any natural number $n > 1$, the unique prime factorization in classical number theory corresponds precisely to its representation as a product of elements from $\{P_k \in \mathcal{P} : k \geq 2\}$.

*Proof.* Let $n > 1$ have the classical prime factorization $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ where $p_i$ are distinct classical primes and $a_i > 0$.

By Theorem 5, each classical prime $p_i$ corresponds to exactly one element $P_{k_i} \in \mathcal{P}$ with $k_i \geq 2$. Therefore:

$$n = P_{k_1}^{a_1} P_{k_2}^{a_2} \cdots P_{k_r}^{a_r}$$

This representation is unique because:

(1) The classical factorization is unique by the Fundamental Theorem of Arithmetic
(2) The correspondence established in Theorem 5 is bijective
(3) Products involving $P_1 = 1$ do not affect the factorization structure since $1^k = 1$ for any $k$

Therefore, the mereological framework preserves the essential uniqueness property while providing the generative foundation through $P_1$. □

This result demonstrates that the mereological framework extends rather than contradicts classical number theory. The inclusion of $P_1 = 1$ provides the mathematical foundation for complete number generation while preserving all structural properties required for unique factorization of integers greater than 1.

The following theorem demonstrates how this classical result emerges naturally within our recursive framework while providing a more ontologically coherent foundation for understanding the relationship between mathematical objects and their structural components.

**Theorem 3.3** (Fundamental Theorem of Arithmetic - Mereological Version)**.** *Every natural number $\geq 1$ can be uniquely expressed as a product of powers of primes $P_n$ for $n \geq 1$, such that the whole $n$ is recursively constructed from irreducible parts via repeated compositional operations (multiplication). This decomposition is unique up to the ordering of parts.*

The classical Fundamental Theorem of Arithmetic exemplifies how mathematical necessity can emerge from structural relationships rather than definitional stipulation. Our mereological reformulation reveals this theorem as expressing a deeper principle about recursive construction and structural decomposition.

*Proof.* From Theorem 5, we established that our mereological definition generates the same sequence of primes (excluding $P_1 = 1$) as the traditional definition. Since the Fundamental Theorem of Arithmetic holds in traditional number theory, and our prime sequence (excluding $P_1$) is identical to the traditional sequence, the theorem holds in our framework as well.

*Note:* The foundational unit $P_1 = 1$ serves as the mereological ground state but is excluded from multiplicative compositions in this theorem. It acts as the identity element for composition, not as a proper factor. Thus, uniqueness of decomposition remains preserved in the structural framework.

Specifically, any natural number greater than 1 can be uniquely expressed as:

$$n = P_2^{a_2} \cdot P_3^{a_3} \cdot \ldots \cdot P_k^{a_k}$$

where each $a_i$ is a non-negative integer, and the representation is unique up to the ordering of the factors. □

This theorem reveals how our framework maintains the essential structural properties of traditional number theory while providing a more ontologically satisfying account of the role of 1.

This result confirms that the classical Fundamental Theorem of Arithmetic is a special case of a deeper mereological principle: every whole in the number system arises from irreducible atomic elements through a unique structure-preserving composition. Unlike classical formulations that rely on external definitions of divisibility, this formulation internalizes primality and composition, aligning with ontological structuralism.

**Definition 3.4** (Unique Decomposability)**.** Every finite composite in the number system has a unique minimal cover of irreducible parts under recursive multiplication.

This reformulation demonstrates how foundational mathematical results can be understood through the lens of recursive construction. The uniqueness of prime factorization reflects the logical necessity inherent in recursive structural processes rather than merely a convenient property of divisibility relationships.

**Definition 3.5** (Minimal Cover)**.** A minimal cover of a composite number $n$ is a multiset of irreducible parts whose product equals $n$, such that no proper subset has this property.

The Unique Decomposability axiom is equivalent to the classical statement of the Fundamental Theorem of Arithmetic within our mereological framework.

3.4. **The Inadequacy of Divisibility-Based Definitions.** The standard pedagogical definition of primes as "numbers having exactly two distinct positive divisors" exemplifies how mathematical practice can obscure structural necessities through ad hoc exceptions. This definition immediately encounters the problematic case of 1, which has only one divisor, leading to various exclusionary formulations that treat 1 as a special exception rather than recognizing its foundational role.

The very need for systematic exceptions indicates that the underlying definition fails to capture essential structural relationships. Our mereological framework resolves this tension by revealing that 1 occupies a categorically different role—not as a failed prime requiring exclusion, but as the foundational element that enables all prime construction.

This definition immediately encounters the case of 1, which requires explicit exclusion. While effective for preserving unique factorization, this exception highlights that the divisor-based approach emphasizes technical convenience over generative structure. Our framework complements it by eliminating the need for such exceptions.

3.4.1. *Logical Necessity and Mathematical Inevitability.* The recursive construction demonstrates how mathematical structures can exhibit logical necessity rather than definitional arbitrariness.

**Theorem 3.6** (Structural Necessity of $P_1 = 1$)**.** *The recursive construction establishes that $P_1 = 1$ is not merely a convenient choice but a structural requirement imposed by the goal of complete natural number generation.*

This exemplifies the broader principle: mathematical content can emerge from logical constraints within formal systems rather than from external stipulations.

**Context and Motivation.** In classical number theory, the exclusion of 1 from the set of primes is typically justified as a technical convention: it preserves the uniqueness of prime factorization and avoids exceptions in theorems. But this view treats definitions as stipulations rather than emergent from structure. Our recursive framework reveals that this "convention" masks a deeper structural necessity: the element 1 must play a unique foundational role in any complete generative system for $\mathbb{N}$.

**Logical Conditions for Generative Systems.** We consider the following necessary conditions for constructing $\mathbb{N}$:

(1) **Completeness:** All $n \in \mathbb{N}$ must be constructible via finite products of irreducible elements.
(2) **Atomicity:** The system must identify irreducible generators—elements that cannot be decomposed into other generators.
(3) **Uniqueness:** Each composite number must have a unique representation (up to order) as a product of irreducibles.

**Definition 3.7** (Constructive Necessity)**.** A mathematical structure exhibits constructive necessity within a formal system $S$ if it arises as the unique solution required to fulfill the system's generative objectives, given its foundational constraints.

**Theorem 3.8** (Necessity of the Multiplicative Identity in Recursive Generation)**.** *Let $\mathcal{P} = \{P_n\}_{n\in\mathbb{N}}$ be a recursively defined sequence of irreducible elements whose finite products generate $\mathbb{N}$. Then the foundational element $P_1$ must satisfy the multiplicative identity property:*

$$\forall n \in \mathbb{N}, \quad P_1 \cdot n = n$$

*Proof.* Let $F \in \mathbb{N}$ denote the foundational generator used to initiate the recursive construction of $\mathbb{N}$. Suppose every $n \in \mathbb{N}$ is expressible as a finite product of irreducibles from $\mathcal{P}$. Let $p \in \mathcal{P}$ be any prime, irreducible under the construction. For $p$ to be generated via $F$, we require $F \cdot p = p$. By the cancellation law in $\mathbb{N}$, this implies $F = 1$. Since every $n$ is built from such $p$, we conclude $F \cdot n = n$ for all $n \in \mathbb{N}$, and the only such $F$ is 1. Therefore, $P_1 = 1$ is not a convention, but a structural requirement. □

**Consequence: Recursive Primality with Foundational Unity.** Enforcing $P_1 = 1$ yields:

(1) Well-founded generation: every number can be constructed from the base case.
(2) Irreducibility clarity: primes emerge as minimal elements that ae non-decomposable.
(3) Factorization uniqueness: representations are deterministic and conflict-free.

**Philosophical Implication.** Apparent definitional choices often reflect deeper logical necessities. The traditional exclusion of 1 from primality represents a profound category error: treating as definitional convention what is actually logical necessity. Recursive analysis reveals these structural requirements.

**Corollary 3.9** (Failure of Recursive Generation without Foundational Unit)**.** *Let* $\mathcal{P}' = \{P_n\}_{n\geq 2} \subset \mathbb{N}$ *be a recursive construction of irreducible elements excluding* $P_1 = 1$*. Then the system fails to satisfy at least one of the following:*

(1) *Completeness of generation over* $\mathbb{N}$
(2) *Uniqueness of factorization*
(3) *Closure under recursive construction*

*Proof.* (1) Without 1, the number 1 cannot be generated, so the system is incomplete.

(2) If 1 is excluded but acts as a neutral factor, every $n$ has infinitely many factorizations: $n = 1 \cdot n = 1^2 \cdot n = \ldots$, violating uniqueness.

(3) Recursive definitions like $P_{n+1} := \min\{m \notin \text{span}(P_1, \ldots, P_n)\}$ fail to initialize without $P_1 = 1$. Thus, recursion collapses. □

**Conclusion.** Excluding $P_1 = 1$ leads to structural failure. Therefore, $P_1 = 1$ is a logically necessary element of any well-defined recursive generative system for $\mathbb{N}$.

## 4. Graph-Theoretic Formalization of the Recursive Structure

**4.1. Directed Graph Representation.** The recursive construction $\mathcal{P}$ admits a natural representation as a directed graph that reveals the structural relationships between natural numbers. This graph-theoretic formalization provides precise characterizations of compositional complexity and connectivity properties within the number system.

**Theorem 4.1** (Prime Generation Graph)**.** *Define the directed graph $G_{\mathcal{P}} = (V, E)$ where:*

1. $V = \mathbb{N}$
2. $E = \{(a, b) \in \mathbb{N}^2 : \exists P_k \in \mathcal{P},\ k \geq 2,\ \text{such that } b = a \cdot P_k\}$

**Theorem 4.2** (Structural Properties)**.** *The graph $G_{\mathcal{P}}$ satisfies:*

1. ***Root Property:*** *in-deg*$(1) = 0$ *and 1 is reachable from no other vertex*
2. ***Prime Characterization:*** *For $p > 1$, $p \in \mathcal{P}$ if and only if in-deg*$(p) = 1$ *with the unique incoming edge from vertex 1*
3. ***Composite Characterization:*** *$n$ is composite if and only if in-deg*$(n) \geq 2$
4. ***Path-Factorization Correspondence:*** *The number of directed paths from 1 to $n$ equals the number of ordered factorizations of $n$ using elements from $\{P_k : k \geq 2\}$*

*Proof. Property 1:* By definition, no edge $(a, 1) \in E$ exists since $1 = a \cdot P_k$ with $P_k \geq 2$ has no solutions in $\mathbb{N}$.

*Property 2:* If $p \in \mathcal{P}$ with $p > 1$, then by recursive construction, $p$ cannot be expressed as $a \cdot b$ with $a, b > 1$ from prior elements. The only way to reach $p$ is via $1 \cdot p = p$, giving exactly one incoming edge. Conversely, if in-deg$(p) = 1$ with edge from 1, then $p$ cannot be a non-trivial product, so $p \in \mathcal{P}$.

*Property 3:* If $n$ is composite, then $n = a \cdot b$ where $a, b \in \text{span}(\mathcal{P})$ and $a, b > 1$. This creates multiple paths to $n$, hence in-deg$(n) \geq 2$. The converse follows similarly.

*Property 4:* Each directed path from 1 to $n$ corresponds to a sequence of multiplications by elements from $\{P_k : k \geq 2\}$, which is precisely an ordered factorization. □

**Theorem 4.3** (Complexity and Distance Properties)**.** *In $G_{\mathcal{P}}$:*

1. ***Shortest Path Length:*** $d(1, n) = \Omega(n)$ *where $\Omega(n)$ is the number of prime factors of $n$ (counted with multiplicity)*
2. ***Diameter Bound:*** $d(1, n) \leq \lfloor \log_2(n) \rfloor$ *for all $n \geq 1$*

(3) ***Path Enumeration:*** *The number of directed paths from 1 to $n$ equals $\tau^*(n)$, the number of ordered factorizations of $n$*

*Proof. Property 1:* Each edge corresponds to multiplication by a prime $P_k \geq 2$, so the shortest path uses exactly $\Omega(n)$ steps.

*Property 2:* Since each multiplication by $P_k \geq 2$ at least doubles the value, reaching $n$ from 1 requires at most $\log_2(n)$ steps.

*Property 3:* Each path corresponds to a sequence of prime multiplications, which defines an ordered factorization. □

This graph representation provides a powerful visualization of the mereological structure of the number system, revealing how primes function as the generative elements of the entire structure.

### 4.2. Algebraic Structure and Lattice Properties.

In the mereological framework, primes establish foundational relationships that determine the structure of the entire number system. These relationships can be formalized as follows:

**Definition 4.4** (Divisibility Subgraph). For each $P_k \in \mathcal{P}$ with $k \geq 2$, define the divisibility subgraph:

$$G_k = \{n \in \mathbb{N} : P_k \mid n\}$$

The intersection of prime influence zones $Z(p) \cap Z(q)$ for distinct primes $p$ and $q$ corresponds to numbers divisible by both $p$ and $q$, forming a substructure of compositeness.

**Theorem 4.5.** *The mereological structure of natural numbers forms a complete lattice under the divisibility relation, with join operation corresponding to least common multiple and meet operation corresponding to greatest common divisor.*

*Proof.* Consider any two natural numbers $a, b \in \mathbb{N}$. Their greatest common divisor $\gcd(a, b)$ represents the largest shared structural component, while their least common multiple $\text{lcm}(a, b)$ represents the smallest number containing both $a$ and $b$ as parts. Since gcd and lcm exist for any pair of natural numbers and satisfy the lattice axioms (commutativity, associativity, absorption), the natural numbers under divisibility form a complete lattice structure. □

This lattice perspective provides a formal foundation for understanding how prime numbers function as structural atoms in the mereological framework.

**Theorem 4.6** (Lattice Structure). *The divisibility relation on $\mathbb{N}$ under the recursive construction forms a complete lattice $(L, \leq)$ where:*

(1) *$a \leq b$ if and only if $a$ divides $b$*
(2) *$join(a, b) = lcm(a, b)$*
(3) *$meet(a, b) = \gcd(a, b)$*
(4) *The atoms of this lattice are precisely $\{P_k : k \geq 2\}$*

*Proof.* The divisibility relation is a partial order on $\mathbb{N}$. For any subset $S \subseteq \mathbb{N}$:

(1) The least upper bound exists as $\text{lcm}(S)$ (when finite) or the appropriate limit
(2) The greatest lower bound exists as $\gcd(S)$

The atoms are elements that cover only 1 in the lattice ordering, which are precisely the elements of $\mathcal{P}$ with index $\geq 2$ by Theorem 10. □

**Definition 4.7** (Graph-Theoretic Highly Composite Numbers)**.** A number $n$ is highly composite in $G_{\mathcal{P}}$ if $\text{in-deg}(n) > \text{in-deg}(k)$ for all $k < n$.

**Theorem 4.8** (Hub Characterization)**.** *Highly composite numbers in $G_{\mathcal{P}}$ satisfy:*

(1) ***Maximal Connectivity:*** *They achieve local maxima in the in-degree function*
(2) ***Factorization Richness:*** *They admit the maximum number of distinct factorizations among numbers of similar magnitude*
(3) ***Structural Centrality:*** *They serve as convergence points for multiple prime multiplication paths*

*Proof.* Property 1 follows directly from the definition. Property 2 follows from the previous theorem since in-degree equals the number of ordered factorizations. Property 3 follows from the graph structure where high in-degree corresponds to centrality in the multiplication network.

## 5. Mereological Extension to the Real Continuum

### 5.1. Systematic Extension of the Recursive Framework.

The recursive construction $\mathcal{P} = \{P_n : n \in \mathbb{N}\}$ with $P_1 = 1$ extends systematically to encompass all real numbers through well-defined algebraic and topological operations. We establish that the mereological structure preserves its foundational properties under these extensions.

**Theorem 5.1** (Integer Extension)**.** *Define the mereological integers as:*

$$\mathbb{Z}_{\mathcal{P}} = \{a - b : a, b \in span(\mathcal{P})\}$$

*Then $\mathbb{Z}_{\mathcal{P}} = \mathbb{Z}$ and every integer inherits a canonical mereological representation.*

*Proof.* Every integer $z \in \mathbb{Z}$ can be written as $z = a - b$ where $a, b \in \mathbb{N}$. By Theorem 3, both $a$ and $b$ are expressible as finite products from $\mathcal{P}$, hence $z \in \mathbb{Z}_{\mathcal{P}}$. Conversely, every element of $\mathbb{Z}_{\mathcal{P}}$ is an integer by construction. The foundational role of $P_1 = 1$ is preserved since $1 \cdot n - 1 \cdot m = n - m$ for any integers. □

**Theorem 5.2** (Rational Extension)**.** *Define the mereological rationals as:*

$$\mathbb{Q}_{\mathcal{P}} = \left\{ \frac{p}{q} : p \in \mathbb{Z}_{\mathcal{P}}, q \in \mathit{span}(\mathcal{P}) \setminus \{0\} \right\}$$

*Then $\mathbb{Q}_{\mathcal{P}} = \mathbb{Q}$ with unique mereological representation for each rational.*

*Proof.* Every rational $r \in \mathbb{Q}$ has the form $r = \frac{p}{q}$ where $p \in \mathbb{Z}, q \in \mathbb{N} \setminus \{0\}$. From Theorem 11, $p \in \mathbb{Z}_{\mathcal{P}}$, and from Theorem 3, $q \in \text{span}(\mathcal{P})$, so $r \in \mathbb{Q}_{\mathcal{P}}$. The representation is unique due to the fundamental property of rational numbers and the uniqueness established in Corollary 2. □

### 5.2. Completion to the Real Continuum.

**Definition 5.3** (Mereological Real Numbers)**.** Define $\mathbb{R}_{\mathcal{P}}$ as the metric completion of $\mathbb{Q}_{\mathcal{P}}$ under the standard absolute value metric.

**Theorem 5.4** (Real Extension and Isomorphism)**.** *The mereological real numbers satisfy:*

1. *$\mathbb{R}_{\mathcal{P}}$ is isometrically isomorphic to $\mathbb{R}$*
2. *Every real number is the limit of a Cauchy sequence from $\mathbb{Q}_{\mathcal{P}}$*
3. *The foundational structure from $P_1 = 1$ extends to the entire continuum*

*Proof.* (1) Since $\mathbb{Q}_{\mathcal{P}} = \mathbb{Q}$ (Theorem 12) and both inherit the same metric from $\mathbb{R}$, their completions are isometrically isomorphic.

(2) This follows from the standard density and completeness properties of rational approximation.

(3) The multiplicative identity property $1 \cdot r = r$ extends to all real numbers, preserving the foundational role of $P_1 = 1$ throughout the extension. □

**Corollary 5.5** (Transcendental Numbers)**.** *Transcendental numbers $\alpha \in \mathbb{R}_{\mathcal{P}}$ are characterized as limits of sequences in $\mathbb{Q}_{\mathcal{P}}$ that cannot be expressed through finite algebraic operations on elements from $\mathcal{P}$.*

### 5.3. Structural Preservation Properties.

**Theorem 5.6** (Mereological Structure Preservation)**.** *Under the extensions $\mathbb{N} \subset \mathbb{Z}_{\mathcal{P}} \subset \mathbb{Q}_{\mathcal{P}} \subset \mathbb{R}_{\mathcal{P}}$:*

(1) ***Foundation Preservation:*** *$P_1 = 1$ maintains its role as multiplicative identity across all extensions*
(2) ***Prime Structure:*** *Elements $\{P_k : k \geq 2\}$ remain irreducible within each extended system*
(3) ***Factorization Coherence:*** *Unique factorization properties extend consistently through the hierarchy*

*Proof.* (1) The multiplicative identity property is preserved under field extensions.

(2) Irreducibility is maintained since no new factorizations are introduced that weren't present in the base system.

(3) The unique factorization domain properties extend naturally through the construction sequence. □

5.4. **Topological Properties.** This paper has established a formal mereological framework for prime number theory that advances both mathematical foundations and philosophical understanding of arithmetic structure. The investigation yields contributions on two interconnected levels: technical mathematical results and foundational conceptual insights.

Beyond these technical results, the framework establishes a fundamental metamathematical principle: apparent definitional choices in mathematical foundations often reflect deeper logical necessities discoverable through constructive analysis. The recursive construction demonstrates that $P_1 = 1$ exhibits structural necessity rather than definitional convention, showing that the traditional exclusion of 1 can be interpreted as treating a structural inevitability as a stipulation of convenience. This analysis exposes how mathematical practice can mask structural requirements through systematic exceptions (such as "primes have exactly two divisors, except 1"), highlighting that divisor-based formulations emphasize convenience over generative structure.

The mereological perspective provides a systematic method for distinguishing genuine structural requirements from arbitrary stipulations, offering new insights into the ontological status of mathematical objects as positions within recursively generated structures rather than entities defined solely through external relations. The extension to real numbers establishes that these foundational principles generalize beyond discrete mathematics while maintaining structural coherence.

**Theorem 5.7** (Topological Properties)**.** *The extended system $\mathbb{R}_{\mathcal{P}}$ exhibits:*

(1) ***Completeness:*** *Every Cauchy sequence in $\mathbb{Q}_{\mathcal{P}}$ converges in $\mathbb{R}_{\mathcal{P}}$*

(2) ***Density:*** $\mathbb{Q}_{\mathcal{P}}$ *is dense in* $\mathbb{R}_{\mathcal{P}}$
(3) ***Separability:*** $\mathbb{R}_{\mathcal{P}}$ *is separable with* $\mathbb{Q}_{\mathcal{P}}$ *as countable dense subset*

*Proof.* These follow from standard properties of metric completion applied to the rational extension $\mathbb{Q}_{\mathcal{P}}$. □

## 6. Conclusion

This paper has developed a formal mereological framework for prime number theory that provides an alternative axiomatization through recursive construction. The key mathematical contributions are:

(1) a rigorous recursive definition that generates the standard prime sequence while incorporating 1 as the foundational element (Definition 2.1, Lemma 2.5),
(2) formal equivalence results demonstrating preservation of all classical number-theoretic properties for n  2 (Theorems 2.4, 2.6),
(3) graph-theoretic formalization revealing structural properties of the recursive number system (Section 4), and
(4) systematic extension to the real continuum with preserved mereological structure (Section 5).

The framework demonstrates that alternative foundational choices can preserve mathematical content while revealing different structural perspectives. The recursive construction shows that 1 can function as a foundational element within a systematic generative approach, complementing rather than contradicting the conventional exclusion of 1 that preserves unique factorization properties. This analysis illustrates how mathematical foundations can admit multiple consistent formulations that emphasize different structural aspects. The mereological perspective provides insights into mathematical objects as positions within recursively generated structures rather than entities defined solely through divisibility properties. The systematic extension to real numbers establishes that these recursive principles generalize beyond discrete mathematics while maintaining structural coherence, suggesting broader applications for recursive construction methods in mathematical foundations. Rather than challenging established mathematical practice, this work contributes to foundational studies by demonstrating how alternative recursive approaches can illuminate structural relationships. The framework reveals the distinction between backward-looking divisibility testing and forward-looking generative construction, offering complementary perspectives on the emergence of mathematical structure through systematic recursive processes.

6.1. **Future Directions.** The recursive framework suggests several avenues for further investigation:

(1) **Foundational Studies:** Investigation of recursive construction principles in other mathematical domains, exploring how generative approaches can complement conventional definitional methods.
(2) **Computational Applications:** Development of algorithms based on recursive generation rather than divisibility testing, potentially offering new approaches to primality testing and number-theoretic computation.
(3) **Mathematical Structuralism:** Further exploration of how alternative axiomatizations can preserve mathematical content while revealing different structural perspectives, contributing to ongoing discussions in philosophy of mathematics.
(4) **Extensions:** Investigation of similar recursive construction principles in other number systems and mathematical structures, including complex numbers and algebraic number fields.

The formal framework developed here provides a foundation for further investigation of recursive mathematical structures and their applications to foundational questions in logic and mathematics, demonstrating that mathematical foundations can admit multiple consistent perspectives that illuminate different aspects of mathematical structure.

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